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# Covering by Random Intervals and One-Dimensional Continuum Percolation

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A brief historical introduction is given to the problem of covering a line by random overlapping intervals. The problem for equal intervals was first solved by Whitworth in the 1890s. A brief resume is given of his solution. The advantages of the present author's approach, which uses a Poisson process, are outlined, and a solution is derived by Laplace transforms. The method of Hammersley for dealing with a stochastic distribution of intervals is described, and a solution can still be derived by Laplace transforms. The asymptotic behavior as the line becomes long is calculated and is related to the one-dimensional continuum percolation problem. It is shown that as long as the mean interval size is finite, the probability of complete coverage decays exponentially, so that the critical percolation probability $p_c = 1$. However, as soon as the mean interval size becomes infinite, the critical percolation probability $p_c$ switches to 0. This is in accord with previous results for a lattice model by Chinese workers, but differs from those of Schulman. A possible reason for the discrepancy is a difference in boundary conditions.

**KEY WORDS:** Continuum percolation; random intervals.

## 1. HISTORICAL INTRODUCTION

For several years during World War 2, I was engaged in radar research for the Admiralty, and was concerned (among other things) with the theory of Poisson processes and random noise. When I returned to Cambridge in 1946 to take up a postgraduate appointment, I brought with me a theoretical problem which had been stimulated by work in this area. Suppose each event in a Poisson process is the left-hand end of an interval $\tau$. Choose any

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section [0, $y$] of the line. What is the probability $z(y)$ that the section is completely covered? I found that this problem could be solved simply and neatly using Laplace transforms.

I needed to know whether anyone had tackled the problem previously, and Herman Bondi (who had been one of my colleagues at the Admiralty) referred me to Harold Jeffreys, whom he described as a mine of information on miscellaneous mathematical problems. Jeffreys immediately thought of the "bicycle wheel problem" which he himself had formulated a few years previously as follows: A man is cycling along a road and passes through a region strewn with tacks; he wishes to know whether one has entered his tire. Because of the traffic, he can only snatch glances at random times. At each glance he covers a fraction $x$ of the wheel. What is the probability that after $n$ glances he has covered the whole wheel? In mathematical terminology: $n$ intervals are placed randomly on a circle, each covering a fraction $x$ of the circle. What is the probability that the circle is completely covered? (Fig. 1).

Jeffrey's drew my attention to a paper published by W. L. Stevens in 1939 in the *Annals of Eugenics*,[1] entitled, "Solution to a Geometrical Problem in Probability," in which his problem was solved. Using a neat combinatorial argument, Stevens found for the probability $F(0)$ of complete coverage

$$F(0)=1-\binom{n}{1}(1-x)^{n-1}+\binom{n}{2}(1-2x)^{n-1}-\binom{n}{3}(1-3x)^{n-1}\cdots \quad (1)$$

the series terminating at the $k$th term, $k$ being the integral part of $1/x$. Stevens also derived a formula for $F(i)$, the probability that there are $i$ gaps on the circle.

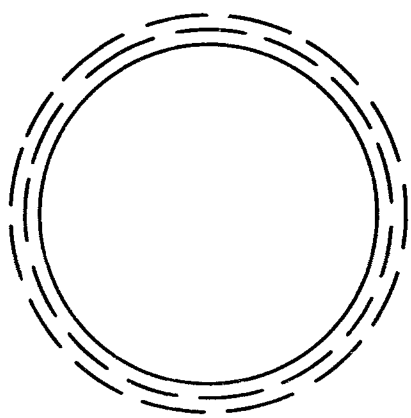

Fig. 1. The bicycle wheel problem.

In 1929, R. A. Fisher published an article[2] entitled, "Tests of significance in harmonic analysis," in which he calculated the probability that the largest interval in the random division of a circle is less than $x$ (Fig. 2). When Steven's solution for $F(0)$ appeared, Fisher noted that it was identical with his, and a moment's reflection is enough to convince one that the two problems are identical. Fisher pointed this out in a note published in 1940.[3]

But surprisingly, Fisher, one of the founders of the modern theory of statistics, was unaware that the distribution of length of the largest interval in the random division of a line had been correctly solved by Whitworth many years before, and was reproduced in his classic book, *Choice and Chance* (solutions to problems 666 and 667 published in 1897).

Problem 666: A line of length $c$ is divided into $n$ segments by $n-1$ random points. Find the chance that no segment is less than a given length $a$, where $c > na$ (say, $c - na = ma$).

Problem 667: In the last question find the chance that $r$ of the segments shall be less than $a$ and $n-r$ greater than $a$.

More precise dating of the solutions will be discussed in the next section.

My own contribution,[5] in 1947, was to deal with the problem as I had formulated it in relation to a Poisson process, and to derive an integral equation which could be solved by Laplace transforms. I was able to find closed-form solutions for $z(y)$, the probability that a section $[0, y]$ of the line is completely covered; $z_k(y)$, the probability that it contains $k$ gaps; and $W(x, y)\,dx$; the probability that the covered portion of the line is

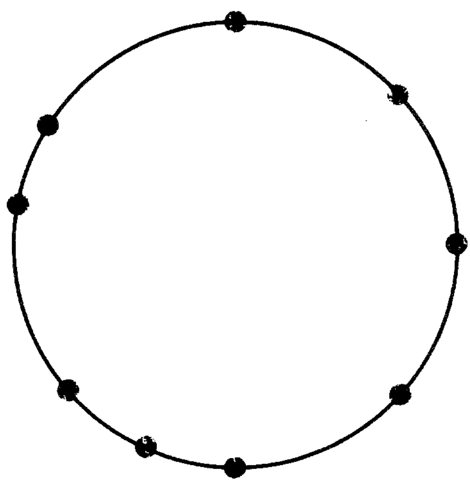

Fig. 2. Random division of a circle.

between $x$ and $x+dx$. Deriving a solution for a Poisson process with a random parameter $\lambda$ (the probability of an event occurring in $[x, x+dx]$ being $\lambda\, dx$) is similar to using a grand partition function in statistical mechanics; the solution for fixed $n$ can then be derived from it by picking out the $n$th term in a suitable expansion.

The Laplace transform solution has the advantage that asymptotic behavior for large $y$ can be calculated by a standard procedure, whereas it is much more difficult to handle a combinatorial formula like (1).

At about this time there was considerable interest in the statistics of particle counters with a finite resolution time $\tau$. Events are divided into two classes, recorded and unrecorded. A recorded event is followed by a dead interval during which any other event which occurs will be unrecorded. A typical example is an $\alpha$-particle counter; a recorded particle causes the chamber to ionize, and no other particle can be recorded until the chamber has deionized. This is a different problem from that considered above and will be called a type 1 counter (Fig. 3a); probability distributions of recorded events can also be solved readily by Laplace transforms.[6] But an alternative type of recorder, relevant, for example, to blood counts, remains dead as long as events follow one another at intervals less than $\tau$. This recorder (type 2, Fig. 3b) gives rise to problems very similar to those considered above in relation to the covering of a line or circle.

In considering the statistics of a blood cell counter, Hammersley generalized this latter problem to the case of a stochastic distribution of intervals $u(\tau)\, d\tau$, and showed how to overcome mathematical difficulties which arise because an interval associated with an event can completely cover the interval associated with an event which follows it. Subsequently, Smith[8] demonstrated that renewal theory could be effectively applied to the problem; he was able to simplify Hammersley's arguments, and to

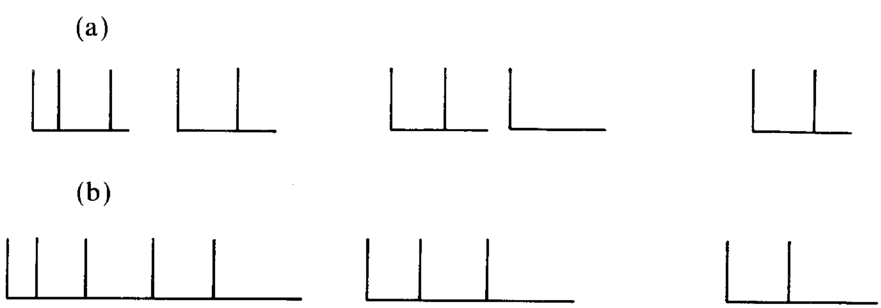


Fig. 3. (a) Type 1 counter, which does not function during the "dead" time following a recorded event. All *recorded* events have the same length. (b) Type 2 counter. Events can overlap and recorded events have different lengths.

remove a restriction which Hammersley had imposed that the distribution $u(\tau)\,d\tau$ must be bounded. Both Hammersley and Smith were concerned with the mean, mean square, and asymptotic forms of the distribution of recorded events.

Other aspects of the covering problem for equal intervals were discussed by Flatto and Konheim,[9] including the expected number of intervals just needed to cover the circle. These authors refer to "a vast body of literature" relating to the problem, but do not give information which could give the reader access to this literature. In the present survey I do not attempt to provide an exhaustive list of papers on the topic, but hope to draw attention to a restricted list of publications from which anyone interested can trace the relevant literature.

A new question first raised by Dvoretsky[10] was that of covering a circle by $n$ intervals whose length $\tau_n$ depends on $n$. How rapidly could $\tau_n$ decay to zero if the probability of covering the circle was still to remain unity? Further work on this topic (together with references to the literature) is contained in a paper by Shepp.[11]

Percolation processes were first introduced by Broadbent and Hammersley[12] in 1957, but no one realized that the overlapping-intervals problem was precisely the problem of continuum percolation in one dimension. When 24 years later Shalitin[13] published a discussion of one-dimensional continuum percolation, he was, not surprisingly, unaware of the above literature, whose discussion had taken place in a different context.

Shalitin was concerned with equal intervals. An interesting question which attracted much attention subsequently is that of a stochastic distribution $u(\tau)\,d\tau$ with a long tail to infinity. How slow must this decay be to ensure a critical percolation probability $p_c$ different from 1, the value for a short-range decay? Lattice models of this problem were considered by Zhang *et al.*,[14] Schulman,[15] Newman and Schulman,[16] and Aizenman and Newman,[17] who concluded that in a power law decay of the form $u(\tau) \sim \tau^{-s}$, $p_c = 1$ for $s > 2$, $p_c < 1$ for $s < 2$, and that the marginal case $s = 2$ needs special attention. One would not *a priori* expect a significant difference in this case between lattice and continuum models. However, we shall see that the matter does require closer examination.

The methods used by workers in percolation theory differ from those used by the earlier statistical authors. It is the aim of the present paper to show that the approach used previously by Domb[5] and Hammersley[7] provides an alternative method of dealing with percolation problems, and can give rise to interesting new results. I shall also endeavor to explain a discrepancy between the conclusions of Zhang *et al.*,[14] and those of Schulman.[15]

## 2. WHITWORTH'S *CHOICE AND CHANCE*

I will preface this section with a few biographical details relating to Whitworth,[18] and will continue with some comments on the different editions of his famous publication, *Choice and Chance*.

William Allen Whitworth was born in 1840, and entered St. John's College as an undergraduate in October 1958. His performance in the Mathematics Tripos was not distinguished—he was 16th wrangler in 1862—but this does not seem to have represented his true ability. While still an undergraduate he was principal editor of the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, started at Cambridge in November 1861. The publication was continued as *The Messenger of Mathematics*; Whitworth remained one of the editors till 1880, and was a frequent contributor.

After leaving Cambridge in 1862 he was successively chief mathematics master at Portarlington School and Rossal School, and professor of mathematics at Queen's College, Liverpool (1862–1864); he was a fellow of St. John's College from 1867 to 1882. At the same time Whitworth followed a second career of distinction in the Church, being ordained deacon in 1865 and priest in 1866. He held appointments as a curate at three churches in Liverpool from 1865 to 1875, and as vicar of two churches in London from 1875 until his death in 1905.

The first edition of *Choice and Chance* was published in 1867 while he was in Liverpool, and was a reproduction of lectures given to ladies in Queen's College Liverpool in the Michaelmas term of 1866. The book was subitled *Two chapters of Arithmetic*, and its aims, as described in the Preface, were modest enough:

> I had already discovered that the usual method of treating questions of selection and arrangement was capable of modification and so great simplification, that the subject might be placed on a purely arithmetical basis; and I deemed that nothing would better serve to furnish the exercise which I desired for my classes, and to elicit and encourage a habit of exact reasoning, than to set before them, and establish as an application of arithmetic, the principles on which such questions of "choice and chance" might be solved

He expressed the hope that this pubication might be of service "in conducing to a more thoughtful study of arithmetic than is common at present; extending the perception and recognition of the important truth, that arithmetic, or the art of counting, demands no more science than good and exact common sense."

Chapter 1 was devoted to "Choice," and was followed by 24 questions; Chapter 2 to "Chance," followed by 20 questions. The questions were all arithmetical in character. An appendix was devoted to "Permutations and

Combinations Treated Algebraically": "In my experience as a teacher I have found the proofs here set forth more intelligible to younger students than those given in the text books in common use." Whitworth here derived a number of standard elementary combinatorial formulas, and ended with a new combinatorial proof of the binomial theorem.

The second edition, published only 3 years later (1870) from St. John's College, Cambridge, added three appendices containing more sophisticated material. Appendix II was devoted to "Distributions" (into different groups or parcels), Appendix III to "Derangements": "a series of propositions are given which are not usually found in text books of algebra. But I can see no reason why examples of such simple propositions... should be excluded from elementary treatises in which more complex but essentially less important theorems find place." Appendix IV was concerned with the celebrated St. Petersburg problem and its background. More than 100 miscellaneous new examples were added.

In the third edition, published in 1878, the material in the appendices was revised and enlarged, and incorporated into the main text. There were now four chapters on "Choice" and four chapters on "Chance," the final, brief eighth chapter carrying the title, "The Geometrical Representation of Chances." The number of examples was increased to 300, and they were divided into different classes. The Preface contained the proclamation, "Questions requiring the application of the Integral Calculus are not included in the book, which only fulfils its title to be an Elementary Treatise."

In the fourth edition, published in 1886, the number of examples grew to 640, and a new chapter in the "Choice" section was added dealing with problems where the order in which gains and losses occur is relevant, e.g., if there is a condition that losses must never exceed gains. A short additional chapter in the "Chance" section entitled, "The Rule of Succession," was devoted to a precise treatment of situations in which the probability of an event is supposed completely unknown, but the results of a number of trials are available. What can now be predicted about future trials?

The fifth and final edition was not published until 1901. But in 1897 there appeared a volume entitled, *DCC Exercises in Choice and Chance*, which provided fairly detailed solutions to the 640 examples of the fourth edition, and to 60 new examples, several of which were concerned with the random division of a line by a number of points. Questions 666 and 667, which were quoted in the previous section, are included among the latter. The preface to the fifth edition, which now contained 1000 examples, described the new category as follows: "A new feature will be recognized in a class of problems which found scarcely any place in former editions; the class which includes investigations into the mean value of the largest part, (or the smallest, or any other in order of magnitude) or of functons of such a part, when a magnitude is divided at random."

It is clear that Whitworth was actively working on this type of problem at the time. Quoting again from the same preface, "The most important addition in the body of the work is the very far-reaching theorem... which enables us to write down at sight the mean value of such functions as $\alpha^3$, $\alpha^3\beta^4$, $\alpha\beta\gamma$ etc. when $\alpha, \beta, \gamma,$... are the parts into which a given magnitude is divided at random. I first published this theorem in a pamphlet in the year 1898." The calculations of quantities of this type given in the *DCC Exercises* volume did not make use of the theorem, and were much longer.

From the above discussion it is clear that the problem with which we are concerned was tackled by Whitworth at some date between 1886 and 1897, most probably close to the latter date.

A surprising feature of the final two volumes is the lack of appreciation of the power of the method of generating functions (g.f.'s). Standard formulas like

$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots n\binom{n}{n}=n2^{n-1} \tag{2}$$

which are trivially established by g.f.'s, are proved by a combinatorial analysis of the terms on each side of the equation.

## 3. WHITWORTH'S SOLUTION

Whitworth divided the line into a number of discrete segments, which would eventually be allowed to become very large. He then used standard combinatorial formulas which he had developed in the text to enumerate various cases outlined in examples 666 and 667 (see Section 1 of the present paper).

I shall retain Whitworth's notation for historical reasons, but shall find it convenient to use generating functions to reproduce his combinatorial formulas. Whitworth assumed that the line of length $c$ was divided into $\omega c$ equal elements. The given length $a$ would then contain $\omega a$ elements. Take a dummy variable $x_1$ to enumerate the possible configurations of the first segment, $x_2$ the second segment,..., $x_n$ the $n$th segment. Then the g.f. which enumerates all configurations in any division of the line by $n-1$ points is

$$F(t, x_1, x_2,..., x_n) = (tx_1 + t^2x_1^2 + \cdots)(tx_2 + t^2x_2^2 + \cdots)\cdots(tx_n + t^2x_n^2 + \cdots) \tag{3}$$

assuming no two points are identical. The total number of segments is $\omega c$, and therefore all possible configurations are enumerated by the coefficient of $t^{\omega c}$ in $F(t; x_1, \ldots, x_n)$. If we need the total number of configurations, we put $x_1 = x_2 \cdots = x_n = 1$ and find the coefficient of $t^{\omega c - n}$ in $(1-t)^{-n}$, which is

$$\binom{\omega c - n + n - 1}{n-1} = \binom{\omega c - 1}{n-1} = \frac{(\omega c - 1)(\omega c - 2)\cdots(\omega c - n + 1)}{(n-1)!} \tag{4}$$

For problem 666 one needs to enumerate all configurations with each of the segments containing $\omega a$ or more elements, and Whitworth realized that this was identical with finding all possible configurations which divide a line of length $c - n\omega a$ into $n$ parts. This is clear from the g.f. approach, since the appropriate enumerator is now

$$\begin{aligned} &t^{\omega a} x_1^{\omega a}(1 + tx_1 + t^2 x_1^2 \cdots)\, t^{\omega a} x_2^{\omega a}(1 + tx_2 + t^2 x_2^2 \cdots) \cdots \\ &\quad \times t^{\omega a} x_n^{\omega a}(1 + tx_n + t^2 x_n^2 + \cdots) \end{aligned} \tag{5}$$

We therefore require the coefficient of $t^{\omega(c-na)}$, i.e., of $t^{\omega na}$ $(ma = c - na)$ in $(1-t)^{-n}$, which is

$$\binom{\omega ma + n - 1}{n-1} = \frac{(\omega ma + n - 1)(\omega ma + n - 2)\cdots(\omega ma + 1)}{(n-1)!} \tag{6}$$

Hence the probability that no segment is less than $a$ is found by taking the quotient of (6) by (4) and is equal to

$$\frac{(\omega ma + n - 1)(\omega ma + n - 2)\cdots(\omega ma + 1)}{(\omega c - 1)(\omega c - 2)\cdots(\omega c - n + 1)} \tag{7}$$

When $\omega$ increases indefinitely, this reduces to

$$(ma/c)^{n-1} \tag{8}$$

For example 667, Whitworth pointed out that all orders of choice of the $r$ segments less than $a$, and the $n - r$ segments greater than $a$, give rise to the same number of configurations, and we can therefore deal with the case in which the $r$ segments are at the beginning and the $n - r$ at the end, and multiply by $\binom{n}{r}$. The enumerating g.f. is then

$$\begin{aligned} &(tx_1 + t^2 x_1^2 + \cdots t^{\omega a - 1} x_1^{\omega a - 1})(tx_2 + t^2 x_2^2 + \cdots t^{\omega a - 1} x_2^{\omega a - 1}) \cdots \\ &\quad \times (tx_r + t^2 x_r^2 + \cdots t^{\omega a - 1} x_r^{\omega a - 1})\, t^{\omega a} x_{r+1}^{\omega a}(1 + tx_{r+1} + t^2 x_{r+1}^2 \cdots) \\ &\quad \times t^{\omega a} x_{r+2}^{\omega a}(1 + tx_{r+2} + t^2 x_{r+2}^2 \cdots)\, t^{\omega a} x_n^{\omega a}(1 + tx_n + t^2 x_n^2 + \cdots) \end{aligned} \tag{9}$$

The total number of configurations is the coefficient of

$$\frac{(1-t^{\omega a-1})^r}{(1-t)^r}(1-t)^{-(n-r)} = (1-t^{\omega a-1})^r (1-t)^{-n} \tag{10}$$

Expanding the first factor by the binomial theorem, we derive the series

$$\begin{aligned}&\binom{n+\omega(m+r)a-r+1}{n-1} - \binom{r}{1}\binom{n+\omega(m+r-1)a-r}{n-1}\\ &\quad+\binom{r}{2}\binom{n+\omega(m+r-2)a-r+1}{n-1}\cdots\\ &\quad\times(-1)^s\binom{n+\omega(m+r-s)a-r+s-1}{n-1}\cdots\end{aligned} \tag{11}$$

In the limit of very large $\omega$ this simplifies very considerably; dividing by (4) and taking the limit, we obtain

$$\begin{aligned}&\left(\frac{m+r}{m+n}\right)^{n-1} - \binom{r}{1}\left(\frac{m+r-1}{m+n}\right)^{n-1} + \binom{r}{2}\left(\frac{m+r-2}{m+n}\right)^{n-1}\cdots\\ &\quad+(-1)^s\left(\frac{m+r-s}{m+n}\right)^{n-1} + \cdots(-1)^r\left(\frac{m}{m+n}\right)^{n-1}\end{aligned} \tag{12}$$

Expression (12) must be multiplied by $\binom{n}{r}$ to obtain the complete solution.

Although (11) looks complicated, the g.f. (10) from which it is derived is quite simple, and the calculation of averages and higher moments can be undertaken by standard routine.

The probability of complete coverage, with which we have been concerned, corresponds to $r=n$, and is given by

$$1-\binom{n}{1}\left(\frac{c-a}{c}\right)^{n-1} + \binom{n}{2}\left(\frac{c-2a}{c}\right)^{n-1}\cdots + (-1)^s\binom{n}{s}\left(\frac{c-sa}{c}\right)^{n-1} + \cdots \tag{13}$$

the series terminating at the last term before $c-sa$ becomes negative.

The solutions given above are the same as those derived later by Fisher[2] and Stevens,[1] with the slight adaptation needed for a problem on a circle rather than on a line.

## 4. USE OF A POISSON PROCESS: EQUAL INTERVALS

The problem to be considered is the following (Fig. 4).

Events occur at random on a line in a Poisson distribution, the probability of an occurrence in $[t, t+dt]$ being $\lambda\, dt$. Each event is the

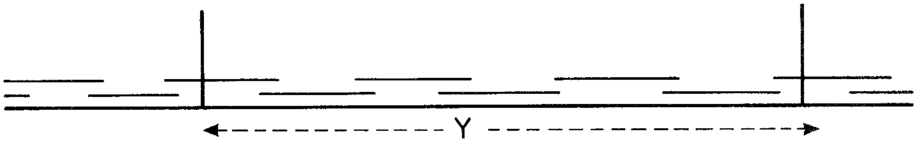


Fig. 4. Random intervals on a line.

left-hand end of an interval $\tau$. Choose any section $[0, y]$ of the line. Calculate the probability $z(y)$ that the section is completely covered.

We divide $z(y)$ into mutually exclusive classes $z(y, \xi)$ in which the last event occurred between $y-\xi$ and $y-\xi-d\xi$. Then if $y>\tau$, $\xi$ cannot be greater than $\tau$ or the section $[0, y]$ would not be covered. Also, $z(y, \xi)$ can be decomposed into three independent contributions: (i) No event occurs in $[y-\xi, y]$; (ii) an event occurs in $[y-\xi-d\xi, y-\xi]$; (iii) the section $[0, y-\xi]$ is covered. Hence, we deduce that

$$z(y)=\int_0^\tau z(y, \xi)\, d\xi=\int_0^\tau \lambda e^{-\lambda\xi} z(y-\xi)\, d\xi \qquad (y>\tau) \tag{14}$$

If $y\leqslant\tau$, we must take into account the additional possibility that an event occurs in $[y-\tau, 0]$, and we easily find that

$$z(y)=\int_0^y \lambda e^{-\lambda\xi} z(y-\xi)\, d\xi+e^{-\lambda y}-e^{-\lambda\tau} \qquad (y\leqslant\tau) \tag{15}$$

Taking Laplace transforms in $y$ in (14) and (15), we derive for the Laplace transform $Z(p)$ of $z(y)$,

$$Z(p)=\frac{p(1-e^{-\lambda\tau})-\lambda e^{-\lambda\tau}(1-e^{-p\tau})}{p+\lambda e^{-(p+\lambda)\tau}} \tag{16}$$

If the denominator is expanded as $[1+(\lambda/p)e^{-(p+\lambda)\tau}]^{-1}$ and the terms are interpreted individually, the combinatorial solution is obtained. If further the solution is broken down into mutually exclusive classes in which exactly $n$ events occur in $[0, y]$, the identity

$$z(y)=\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} f_n(y) \tag{17}$$

can be deduced, where $f_n(y)$ is the probability for $n$ events. In this way the solution of Whitworth, Fisher, and Stevens can be simply derived.

But if we are interested in large $y/\tau$, the asymptotic behavior of $z(y)$

is determined by the zeros of the denominator of (16), i.e., by solutions $-\gamma_s$ of

$$q+\beta e^{-(\beta+q)}=0 \qquad (q=p\tau,\ \beta=\lambda\tau) \tag{18}$$

There is only one real root, $-\gamma$, which dominates the asymptotic behavior, the complex roots providing transients which rapidly decay. $\gamma$ is the solution other than $\beta$ of the equation

$$xe^{-x}=\beta e^{-\beta} \tag{19}$$

(see Fig. 5). We then find the asymptotic solution

$$z(y)-\frac{e^{-\beta}(\beta-\gamma)}{\gamma(1-\gamma)}e^{-\gamma v} \qquad (y=v\tau) \tag{20}$$

When $\beta$ is large (high density of events), $\gamma$ is small, and when $\beta$ is small, $\gamma$ is large. The probability of an infinite cluster in a one-dimensional percolating system is zero; Eq. (20) describes the approach to zero as a finite system grows large.

The calculation for $z_k(y)$ follows similar lines. The integral equation is now

$$\begin{aligned} z_k(y)&=\int_0^y \lambda e^{-\lambda\xi} z_k(y-\xi)\,d\xi \qquad (\xi\leqslant\tau)\\ z_k(y)&=\int_0^y \lambda e^{-\lambda\xi} z_{k-1}(y-\xi)\,d\xi \qquad (\xi>\tau)\end{aligned} \tag{21}$$

with special treatment for $k=1$. Taking Laplace transforms, we find

$$Z_k(p)=\frac{\lambda e^{-\tau(p+\lambda)}}{p+\lambda e^{-\tau(p+\lambda)}}Z_{k-1}(p)=\left(\frac{\lambda e^{-\tau(p+\lambda)}}{p+\lambda e^{-\tau(p+\lambda)}}\right)^{k-1} Z_1(p) \tag{22}$$

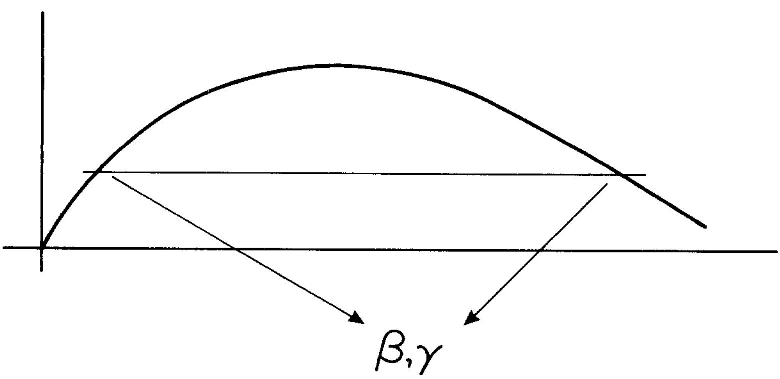


Fig. 5. Solution of $xe^{-x}=\beta e^{-\beta}$ giving asymptotic decay.

From this it can be deduced that the asymptotic distribution of clusters is normal with mean $v\beta e^{-\beta}$ and variance $v[\beta e^{-\beta} - 2\beta^2 e^{-2\beta}]$.

The calculation $W(x, y)\,dx$ is more complicated, and the distribution contains $\delta$-function terms corresponding to various discrete probabilities. The moments of the distribution can be calculated in a straightforward manner. For example,

$$\begin{aligned}\langle x \rangle &= y(1 - e^{-\beta})\\ \langle x^2 \rangle &= y^2(1 - e^{-\beta}) - e^{-\beta}(y^2 - 2y/\lambda + 2/\lambda^2)\\ &\quad + e^{-2\beta}[(y-\tau)^2 - 2(y-\tau)/\lambda + 2\lambda^2]\end{aligned} \tag{23}$$

## 5. STOCHASTIC DISTRIBUTION OF INTERVALS

When the intervals are not all equal the previous method breaks down because an early event can overlap a later one (Fig. 6). The behavior at the point $y$ is no longer dependent only on the latest event at $y - \xi_1$, but all previous events at $y - \xi_1$, $y - \xi_1 - \xi_2$,..., must be considered. The way in which to deal with this new situation was demonstrated by Hammersley,[7] who was interested in the statistics of blood cell counters; I shall adapt his method to the percolation problem (for a renewal theory approach see Smith.[8]

Assume a probability distribution of intervals $u(\tau)\,d\tau$, and divide $z(y)$ into mutually exclusive classes as follows:

$$z(y) = z(y; \xi_1) + z(y; \xi_1, \xi_2) + z(y; \xi_1, \xi_2, \xi_3) + \cdots + z_0(y) \tag{24}$$

where $z(y; \xi_1)$ represents the class in which the point $y$ is covered by the last event at $y - \xi_1$, $z(y; \xi_1, \xi_2)$ represents the class in which the point is *not covered* by the *last event* at $y - \xi_1$, but is covered by the *last but one* at $y - \xi_1 - \xi_2$; $z(y; \xi_1, \xi_2, \xi_3)$ represents the class in which the point $y$ is *not covered* by the *last two events*, but is covered by the *last but two* at

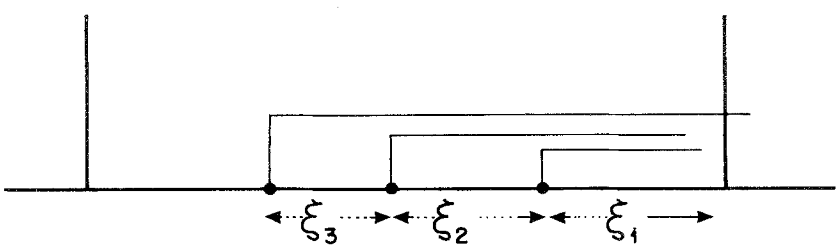


Fig. 6. The first event overlaps the next two.

$y-\xi_1-\xi_2-\xi_3$; $z_0(y)$ represents the class in which *no covering event* occurs in $[0, y]$ but the point $y$ is covered by an event occurring *before*. Write

$$U(\tau)=\int_0^\tau u(t)\,dt \tag{25}$$

which represents the probability of an interval of length $\leqslant\tau$; $1-U(\tau)$ then represents the probability of an interval $>\tau$. It is easy to derive the following relations (Fig. 6):

$$z(y;\xi_1)=\int_0^y \lambda e^{-\lambda\xi}[1-U(\xi_1)]\,z(y-\xi_1)\,d\xi \qquad (0<\xi_1<y)$$

$$z(y;\xi_1,\xi_2)=\iint \lambda e^{-\lambda\xi_1}U(\xi_1)\,d\xi_1\,\lambda e^{-\lambda\xi_2}[1-U(\xi_1+\xi_2)]\,d\xi_2$$
$$\times z(y-\xi_1-\xi_2) \qquad (0<\xi_1,\quad \xi_2<y,\quad \xi_1+\xi_2<y)$$

$$z(y;\xi_1,\xi_2,\xi_2)=\iiint \lambda e^{-\lambda\xi_1}U(\xi_1)\,d\xi_1\,\lambda e^{-\lambda\xi_2}U(\xi_2)\,d\xi_2\,\lambda e^{-\lambda\xi_3}$$
$$\times[1-U(\xi_1+\xi_2+\xi_3)]\,d\xi_3\,z(y-\xi_1-\xi_2-\xi_3)$$
$$(0<\xi_1,\xi_2,\xi_3<y,\quad \xi_1+\xi_2+\xi_3<y) \tag{26}$$

To see the structure of these relations, it is convenient to transform to new variables,

$$\eta_1=\xi_1, \qquad \eta_2=\xi_1+\xi_2, \qquad \eta_3=\xi_1+\xi_2+\xi_3,\ldots \tag{27}$$

so that the limits of integration in the new variables are

$$0<\eta_1<\eta_2<\eta_3\cdots<y \tag{28}$$

We then find

$$z(y;\eta_1)=\int_0^y \lambda e^{-\lambda\eta_1}[1-U(\eta_1)]\,z(y-\eta_1)\,d\eta_1$$

$$z(y;\eta_1,\eta_2)=\iint \lambda^2 U(\eta_1)\,d\eta_1\,e^{-\lambda\eta_2}[1-U(\eta_2)]\,z(y-\eta_2)\,d\eta_2 \tag{29}$$

$$z(y;\eta_1,\eta_2,\eta_3)=\iiint \lambda^3 U(\eta_1)\,d\eta_1\,U(\eta_2)\,d\eta_2\,e^{-\lambda\eta_3}[1-U(\eta_3)]\,z(y-\eta_3)\,d\eta_3$$

The integration in $\eta_1$ in $z(y;\eta_1,\eta_2)$ yields a function of $\eta_2$. Similarly, the integrations of $\eta_1, \eta_2$ in $z(y; \eta_1, \eta_2, \eta_3)$ yield a function of $\eta_3$. The structure of Eq. (24) is therefore

$$z(y) = \int_0^y v(\eta)\, z(y-\eta)\, d\eta + z_0(y) \tag{30}$$

which is still of the form amenable to Laplace transforms. The function $v(\eta)$ can be calculated by summing the successive contributions in (29).

However, I shall use a shortcut to evaluating $v(\eta)$ by considering a related problem, the probability $\zeta(y)$ that the point $y$ is *covered* by an event occurring in $[0, y]$. We can decompose $\zeta(y)$ in a similar manner to (24)–(29) and we obtain the same integrals without the $z(y-\eta)$ factors, i.e.,

$$\zeta(y) = \int_0^y v(\eta)\, d\eta \tag{31}$$

But the probability $1-\zeta(y)$ that the point $y$ is *not covered* by an event occurring in $[0, y]$ was calculated in an elementary manner by Hammersley[7] to be

$$\exp\left[-\lambda y + \lambda \int_0^y U(t)\, dt\right] \tag{32}$$

The derivation is straightforward. Let us call an event which occurs in $[0, y]$ and covers the point $y$ a *covering event*. The probability that a covering event does not occur in the interval $[y-\xi-d\xi, y-\xi]$ is

$$\exp\{-\lambda[1-U(\xi)]\, d\xi\} \tag{33}$$

But all such intervals from $\xi = 0$ to $\xi = y$ are independent. Hence the probability that *no* covering event occurs in $[0, y]$ is the product of factors of type (33) from $\xi = 0$ to $\xi = y$, and this leads directly to (32). Hence we can derive $v(y)$ by differentiating (31),

$$v(y) = \lambda[\exp(-\lambda y)][1-U(y)] \exp\left[\lambda \int_0^y U(t)\, dt\right] \tag{34}$$

On examining (34) and comparing with (28) and (29), it is not difficult to see how the formula could be derived directly, the successive terms in (29) corresponding to successive terms in the expansion of $\exp[\lambda \int_0^y U(t)\, dt]$.

It is convenient to introduce a function $\bar{U}(y)$ which is the complement of $U(y)$,

$$U(y) + \bar{U}(y) = 1 \tag{35}$$

Relations (32) and (34) assume a simplified form in terms of $\bar{U}(y)$ as follows:

$$1 - \zeta(y) = \exp\left[ -\lambda \int_0^y \bar{U}(t)\, dt \right] \tag{36}$$

$$v(y) = \lambda \bar{U}(y) \exp\left\{ \lambda \left[ \int_0^y \bar{U}(t)\, dt \right] \right\} \tag{37}$$

For a distribution $u(\tau)\, d\tau$ which is zero for $\tau \geqslant \tau_0$, $\bar{U}(t)$ is also zero for $\tau \geqslant \tau_0$; for a long-range distribution, $\bar{U}(t)$ provides a direct representation of the tail.

The solution of (30) by Laplace transforms is very simple in principle, and gives for the Laplace transform $Z(p)$ of $z(y)$

$$Z(p) = \frac{Z_0(p)}{1 - V(p)} \tag{38}$$

where $V(p)$ is the Laplace transform of $v(y)$. As in Section 4, the asymptotic behavior of $z(y)$ is determined by the roots of the denominator of (38), and we shall find close parallels to the behavior for equal intervals.

## 6. DISTRIBUTIONS WITH A FINITE MEAN VALUE

It is important to discuss the general behavior of the function $V(p)$ as $p$ decreases from $+\infty$ through zero to $-\infty$. First note that $v(y)$ is positive for all $y$. Hence

$$V(p) = \int_0^\infty v(y) e^{-py}\, dy \tag{39}$$

increases monotonically as $p$ decreases. Thus, there can be only one real root of the equation $V(p) = 1$.

I illustrate this behavior by reconsidering the case of equal intervals, for which

$$u(t) = \delta(t - \tau) \tag{40}$$

$$v(y) = \lambda e^{-\lambda y}, \qquad y \leqslant \tau$$
$$= 0 \qquad y > \tau \tag{41}$$

$$V(p) = \frac{\lambda}{p+\lambda} [1 - e^{-\tau(p+\lambda)}] \tag{42}$$

For large, positive $p$, $V(p)$ is small; as $p$ decreases to zero, $V(p)$ rises to $(1 - e^{-\lambda\tau})$; and at $p = 0$, it is therefore less than 1; for negative $p$, it continues its steady increase, becoming 1 at a unique negative value $-\gamma/\tau$; it then increases exponentially for large negative $p$.

Let us now consider a general distribution with a finite cutoff $\tau_0$. From (37) we see that $v(y)$ is zero for $y > \tau_0$. The general pattern of behavior is similar to that for equal intervals, the value for $p = 0$ being given, from (39), by

$$V(0) = \int_0^\infty v(y)\, dy \tag{43}$$

Using (31) and (36), we find that

$$V(0) = 1 - \exp - \lambda \left[ \int_0^\infty \bar{U}(t)\, dt \right] \tag{44}$$

But

$$\int_0^\infty \bar{U}(t)\, dt = [t\bar{U}(t)]_0^\infty + \int_0^\infty tu(t)\, dt = \bar{\tau} \tag{45}$$

which is the average length of interval. Therefore

$$V(0) = 1 - \exp(-\lambda\bar{\tau}) \tag{46}$$

which is again less than 1. Hence $V(p)$ reaches the value 1 for a negative value of $p = -\bar{\gamma}/\bar{\tau}$, and by analogy with (2) the asymptotic behavior of $z(y)$ is an asymptotic decay, $\exp(-\bar{\gamma}y/\bar{\tau})$. The probability of the line $[0, y]$ being covered tends to zero from large $y$, i.e., there is no percolating cluster.

Now consider a distribution with a long tail of the form

$$u(\tau) \sim A/\tau^s \tag{47}$$

Then

$$\bar{U}(y) = \int_y^\infty u(\tau)\, d\tau \sim A/(s-1)\, y^{s-1} \tag{48}$$

Reverting to Eq. (45), the integral on the left-hand side exists if $s>2$, $\bar{\tau}$ is defined, and the equation remains valid. Hence the argument of the previous section can be repeated, and there is no percolating cluster.

The argument can be extended to a distribution of the form

$$u(\tau) \sim A/\tau^2(\ln \tau)^s \qquad (s>1) \tag{49}$$

for which the integral of $tu(t)$ converges to give a finite mean value $\bar{\tau}$. We now have

$$\bar{U}(y) \sim A/\tau(\ln \tau)^s \tag{50}$$

and Eq. (45) is still valid. Again there is no percolating cluster for large $y$. The argument applies equally for

$$u(\tau) \sim \frac{A}{\tau^2(\ln \tau)(\ln \ln \tau)^s}, \quad \frac{A}{\tau^2 \ln \tau(\ln \ln \tau)(\ln \ln \ln \tau)^s}, \ldots \quad (s>1) \tag{51}$$

the general conclusion being that as long as the mean interval of the distribution is finite, $z(y)$ decays exponentially for large $y$.

## 7. DISTRIBUTIONS WITH AN INFINITE MEAN VALUE

For a distribution $u(\tau)$ for which the integral of $tu(t)$ does not converge, i.e., for which $\bar{\tau}$ becomes infinite, the argument of the previous section would indicate that $V(0)$, which is equal to $1-\exp(-\lambda\bar{\tau})$, becomes equal to 1. Hence, from (38) the dominating term in the asymptotic behavior of $z(y)$ will no longer be an exponential decay, but a constant. Therefore the system will now have a percolating cluster.

We can use the argument of Section 5 to specify in more detail what happens. Consider the probability that the point $y$ is not covered by an event which has occurred in $[-y_0, 0]$. Using Eq. (33), we see that this probability is given by

$$\exp\left[-y\int_y^{y+y_0} \bar{U}(\xi)\, d\xi\right] \tag{52}$$

But for any of the distributions of the previous section for which $\bar{\tau}$ is infinite [(47) with $s \leqslant 2$; (49) and (51) with $s \leqslant 1$] the integral of $\bar{U}(\xi)$ diverges, and by choosing $y_0$ sufficiently large, (52) can be made as small as we please. Hence there is probability 1 that the point $y$ is covered by an event occurring before 0, i.e., that the interval $[0, y]$ is completely covered by such an event. This corresponds to a percolating cluster.

We therefore find that with such distributions percolation occurs however small the value of $\lambda$, so that the system becomes critical however small the percolation probability.

## 8. PREVIOUS RESULTS, LATTICE MODELS

The result of the preceding section was derived independently by Hall[(22)] using an alternative method.[3] Other previous work has been concerned largely with lattice models. Zhang *et al.*,[(14)] who used a transfer matrix method to deal with long-range bond percolation in one dimension, found that, for a distribution of form (47), the critical concentration drops suddenly from 1 to 0 when $s=2$. This conclusion was challenged by Schulman,[(15)] who derived a nonzero critical concentration when $s=2$ using a Monte Carlo method.

There *can* be differences of significance between lattice and continuous percolation models, as was demonstrated by Hall.[(22)] However, I should like to suggest that the origin of the above discrepancy may lie in a difference of boundary conditions. For continuous percolation it is reasonable to consider a Poisson process going on indefinitely, to select an arbitrary section $[0, y]$, and examine whether it is covered or not. This treatment would be somewhat analogous to a one-dimensional lattice percolation problem with a cyclic boundary condition for which the partition function is given by the sum of the $n$th powers of the eigenvalues of a transfer matrix.

It is more usual in lattice problems to deal with a finite system with edges. In standard problems with short-range effects, the two sets of boundary conditions give rise to the same bulk behavior, and differ only in their surface contributions. However, when long-range effects are present, one can no longer separate bulk and surface contributions, and the two problems differ more fundamentally.

It should be possible to devise a continuum analogue of the edge boundary condition problem. A more fundamental approach would be to adapt the method used here for a continuum model to the needs of a lattice system in which generating functions would replace Laplace transforms. I shall endeavor to explore this approach in the near future.

## 9. CONCLUSIONS

The basic result of this work is a simple one: as long as the mean value of the interval length is finite, a one-dimensional percolation model does

[3] I am grateful to a referee for drawing my attention to this paper.

not exhibit critical behavior. I have extended the probability distribution of intervals from $1/\tau^s$, to $1/\tau(\ln \tau)^s$, $1/\tau^2(\ln \tau)(\ln \ln \tau)^s$...., and have found that critical behavior occurs for all the latter distributions when and only when $s \leqslant 2$ for the power law and $s \leqslant 1$ for the rest of the chain. I have given a demonstration that there is critical behavior whenever the mean interval length is infinite.

A similar conclusion that there is no critical behavior when the mean force range is finite has been drawn by Ruelle[19] for the one-dimensional Ising model with long-range forces. But the marginal cases seem to give rise to much more interest and difficulty in the Ising problem.[20,21]

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